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NWC TP 5641

ADB013459

A Continuous Stochastic Model for Penetration Problems

by
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Systems Engineering and Synthesis Division
Weapons Department

JUNE 1976

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Naval Weapons Center

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FOREWORD

A current task to develop procedures for allocating funds between existing system improvement and future system development depends strongly on attrition of strike aircraft as a function of weapons system design and tactical use. This report documents a simple penetration model to improve attrition estimates used in weapons selection models such as SABRE MIX TAC RESOURCER or NAVMOR.

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9 June 1976

NWC Technical Publication 5841

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NWC-TP-5841	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A Continuous Stochastic Model for Penetration Problems.		5. TYPE OF REPORT & PERIOD COVERED Penetration analysis September 1975–May 1976 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Frank C. Reed		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Weapons Center China Lake, Calif. 93555		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 63605N and 62332N, W1175 and WF32311, WF32311005, 151970 and P51104
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Weapons Center China Lake, Calif. 93555		12. REPORT DATE June 1976 13. NUMBER OF PAGES 58
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) Technical publication Sep 75– May 76,		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Distribution limited to U.S. Government agencies only; test and evaluation; 9 June 1976. Other requests for this document must be referred to the Naval Weapons Center.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

WF32-311   WF32-311-005

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Penetration model
Penetration aids
Stochastic models
Nonhomogeneous penetration groups

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

See back of form.

DD FORM 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601

(U) *A Continuous Stochastic Model for Penetration Problems*, by Frank C. Reed. China Lake, Calif., Naval Weapons Center, June 1976. 58 pp. (NWC TP 5841, publication UNCLASSIFIED.)

(U) This report presents a simple method of analysing penetration problems using a combination of continuous stochastic processes and expected value modeling. Groups of different types of penetrators together with nonhomogeneous, mixed defenses may be handled by the model.

CONTENTS

Section 1. INTRODUCTION

This report is concerned with the problem of estimating the survival
of elements of a penetrating attack force using a variety of penetration
aids. This problem has a long history, having been investigated by a
number of aircraft, research, and government organizations. Most
approaches to the problem are based on the fact that the status of the
penetrating force depends on a time sequence of events with the property
that each event depends upon the history of previous events, all actions
that have been taken, and random outcomes associated with these events
and actions.

The level of detail in the modeling usually dictates the proper
approach to the problem. Simple formulations may be handled using
elementary methods of probability. Complex versions of the problem
require either Monte Carlo or Markov Chain solutions. Both Monte Carlo
and Markov Chain approaches keep track of changes in the state of the
penetration process as discrete time events and random outcomes based
on these events and the current state of the process occur. The
Markov Chain solution actually requires discrete time updating of the
probability distribution defined on the state space. There are certain
size state space problems for which the Markov Chain approach is pre-
ferred over the Monte Carlo approach. Generally speaking, if the size
of the state space is large, Monte Carlo methods are preferred. In
penetration studies it is not hard to come up with problems that strain
the capacity of computers and the budgets of those performing the studies.

This report offers another method of handling penetration problems.
Penetration problems are formulated as continuous stochastic processes
using concepts from the area of pure death processes. Whereas time is
the natural parameter of interest in death processes, depth of penetra-
tion is the most suitable parameter to use in penetration problems. The
event/probability updating concept in discrete models is replaced by a
penetration interval/probability updating concept in continuous models.

In this modeling the solution of stochastic differential equations
giving the probability distribution of survivors as a function of
penetration depth is the primary tool of analysis. In some cases the
stochastic differential equations describing the evolution of the
penetration process may be solved in closed form. Even in the most
complex situations numerical procedures for solving the appropriate
stochastic differential equations are conceptually easy to implement;
however, computation can still be a problem. This report presents a
method of handling the complexities of synergistic penetration aid
effects while maintaining a reasonable level of computational effort.

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This report consists of four sections and one appendix in addition
to the introduction. Section 2 discusses simple discrete penetration
models and introduces a discrete approximation which has a well-defined
continuous analogue which is introduced in Section 3. Numerical calcu-
lations which compare the discrete models with the discrete approximation
introduced in Section 2 are found in Appendix A.

Section 3 discusses the basic stochastic processes involved and
presents general model structures. Model complexity is discussed and
various methods of solving the stochastic differential equations associated
with the penetration problem are given. Generally speaking, one should
be able to put together solutions not requiring numerical integration
of the differential equations. The general model for handling penetra-
tors of different types is presented, and statistical independence of
the penetrator types is discussed.

Section 4 is devoted to the calculation of the kill potential that
the defense can bring to bear on a penetration force. It is assumed
that, at the very least, the attributes of a single site against a single
penetrator are known. This data is then input to a simple calculation
to yield cumulative kill potential for a nonhomogeneous defense. The
defense is also allowed to restrict firings to those having an acceptable
yield in terms of kill probability. Another important consideration
in defense capability is the allocation of defensive firings to particu-
lar types of penetrators and the fact that the defense may make errors
in the classification process. If data of a single site as a function
of force size is known, this data may be input to the model directly.

Section 5 is devoted to the effects of penetration aids and tactics.
In this section cumulative kill potential calculations are modified to
account for jinking and/or maneuvering, ECM, defense suppression, self-
defense, and standoff tactics. Since the model is structured to handle
nonhomogeneous penetration forces, it is a simple matter to include
decoys in the model. The analysis of synergistic effects in penetration
aid studies can lead to extremely large state space problems. An iterative
procedure for including synergistic effects while maintaining a reasonable
size state space is discussed.

Section 2. SIMPLE DISCRETE PENETRATION MODELING

In this section we want to discuss simple discrete penetration models. We also want to consider a discrete approximation to these discrete models which has a continuous analogue. This continuous model is derived from basic principles in Section 3. We assume that, with each defense site, there is a maximum number, M, of independent defense/penetrator engagement opportunities which the defense can mount against a penetrating force. The parameter M reflects all the limitations that are imposed on the defense site by recycle times, penetration time in envelope, switching times, ammunition limitations, and reload times. For now, we assume each engagement has a probability of kill of P_k.

An engagement may imply the launch of a salvo of some number of missiles or the use of some nominal number of AAA rounds and then P_k is well defined with respect to the definition of engagement for a given defense site. The most common definition of engagement will be the launch of a single missile or the firing of sufficient AAA rounds to yield a prescribed kill probability, P_k. In the latter instance, a common situation is $M = 1$ and then P_k is a consequence of continuous fire.

2.1 PENETRATOR GROUP SIZE $N \geq M$

When N, the number in the penetrating group is at least as large as M, one is led to a simple solution for the number of penetrators killed. In this case, the defense might as well utilize all M opportunities. The probability that i out of the M penetrators engaged are killed is

$$P(i) = \binom{M}{i} P_k^i (1 - P_k)^{M-i} \qquad i = 0,1,\cdots, M \qquad (2.1)$$

with expectation

$$E = MP_k \qquad (2.2)$$

Frequently one wants a definition of defense capability that is independent of the size of the penetration force. A common measure of defense capability is the product of engagement opportunities and kill probability P_k defined to be the kill potential,

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$$H = MP_k \tag{2.3}$$

In this particular instance when $N \geq M$, $E = H$. The kill potential of a site is a particularly simple measure of defense site effectiveness and it arises naturally in certain discrete approximations to the binomial distribution Equation 2.1.

In fact, if $M \to \infty$ and $P_k \to 0$ in such a way that $MP_k = H$, then Equation 2.1 may be approximated by the Poisson distribution giving

$$P(i) \cong e^{-H} \frac{H^i}{i!} \qquad\qquad i = 0,1,\cdots, \tag{2.4}$$

The implication is that $N \to \infty$ since $N \geq M$ and this approximation will be valid for all i only in those situations in which N and M are both large and P_k is small.

2.2 PENETRATOR GROUP SIZE $N \leq M$

When N is less than M, we want to consider two possible defensive actions. One possibility is to allocate multiple simultaneous engagements (MSE) to a single penetrator. Salvo and ripple firing are examples of allocating more than one engagement at a time to a single penetrator. The second possibility is to use the M encounters in such a way that an encounter is always allocated to a penetrator alive, not currently under attack. Shoot-look-shoot (SLS) firing doctrines fall into this category.

Suppose, for example, $N = 1$, $M = 2$ and there is sufficient time to employ an SLS firing doctrine. The probability of kill of the penetrator is

$$P = P_k + (1 - P_k) \, P_k$$

and the expected number of kills is

$$E = P = 2P_k - P_k^2$$

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Since the kill potential is

$$H = 2P_k \; ,$$

$$E \leq H$$

In an SLS firing doctrine it is possible to kill the penetrator with a single engagement. If an engagement corresponds to the use of a weapon, the expected weapons usage is

$$E_w = 1 \, P_k + 2(1 - P_k) = 2 - P_k$$

For an MSE firing doctrine we have the corresponding results designated by an asterisk,

$$P* = 1 - (1 - P_k)^2 = 2 \, P_k - P_k^2 = P$$

$$E* = P* = P = E$$

with

$$E* \leq H$$

In this case, however, we always use two weapons, so

$$E_w^* = 2 \geq 2 - P_k = E_w$$

More generally, for an SLS firing doctrine the probability that i penetrators are killed is

$$P(i) = \binom{M}{i} P_k^i (1 - P_k)^{M-i} \qquad i = 0,1,\cdots,N-1$$

$$P(N) = \sum_{i=N}^{M} \binom{M}{i} P_k^i (1 - P_k)^{M-i}$$

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In this case,

$$E = \sum_{i=0}^{N-1} i \binom{M}{i} P_k^i (1 - P_k)^{M-i} + N \sum_{i=N}^{M} \binom{M}{i} P_k^i (1 - P_k)^{M-i}$$

$$\leq \sum_{i=0}^{M} i \binom{M}{i} P_k^i (1 - P_k)^{M-i} = M P_k = H$$

So that an SLS firing doctrine leads to

$$E \leq H$$

whenever $N \leq M$. The expected weapon usage is

$$E_w = \sum_{r=0}^{M-N-1} (N + r) \binom{N+r-1}{r} P_k^N (1 - P_k)^r$$

$$+ M \sum_{r=M-N}^{\infty} \binom{N+r-1}{r} P_k^N (1 - P_k)^r$$

$$\leq N + \sum_{r=0}^{\infty} r \binom{N+r-1}{r} P_k^N (1 - P_k)^r$$

$$= N + N \frac{(1 - P_k)}{P_k} = \frac{N}{P_k}$$

Clearly $E_w \leq M$, so

$$E_w \leq \min\left[\frac{N}{P_k}, \quad M\right]$$

For an MSE firing doctrine we assume M/N is an integer and this number of engagements is placed on each of the N penetrators. Immediately

$$E_w^* = M \geq E_w$$

with strict inequality whenever

$$N < M P_k$$

The probability that i penetrators are killed is

$$P^*(i) = \binom{N}{i}\left(1 - (1 - P_k)^{M/N}\right)^i (1 - P_k)^{M/N(N-i)}$$

with the expected number killed given by

$$E^* = N\left(1 - (1 - P_k)^{M/N}\right)$$

One may show that E^* is an increasing function of N, so

$$E^* \leq M\left(1 - (1 - P_k)^{M/N}\right) = M P_k = H$$

It is fairly easy to see that the expected kills in an SLS firing doctrine will exceed those obtainable from an MSE firing doctrine. Suppose the MSE doctrine allocates n_i engagements to the ith penetrator and an SLS doctrine is used on each penetrator, stopping if the penetrator is killed or if n_i engagements are utilized. For the ith penetrator the probability of kill is

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$$P = P_k + q_k\, P_k + \cdots + q_k^{\,n_i-1}\, P_k$$

$$= P_k \frac{\left(1 - q_k^{\,n_i}\right)}{1 - q_k} = 1 - q_k^{\,n_i}$$

which is the probability of kill of the salvo of size n_i in the MSE firing doctrine. In this modified SLS doctrine, providing $P_k > 0$, there is positive probability that not all n_i engagements will be used for all i. It is also clear that, providing $P_k < 1$, there is positive probability that at least one penetrator survives. If δ is the probability that not all engagements are used and not all penetrators are killed, then

$$E \geq E^* + \delta\, P_k > E^*$$

since in an SLS firing doctrine the remaining engagements could be allocated to any survivor.

2.3 A POISSON APPROXIMATION

It was observed in 2.1 that under certain instances the Poisson distribution may be used as an approximation to the binomial. In general, M may be large, but P_k may or may not be large. Generally,

$$\binom{M}{i} P_k^i (1 - P_k)^{M-i} \cong e^{-MP_k} \frac{(M\,P_k)^i}{i!}$$

for M (depending on i) large enough, regardless of P_k. If, moreover, $N \ll M$, then we are concerned with the accuracy of the above approximation for $i = 0, 1, \cdots, N$.

These comments suggest the following approximation for the binomial probabilities under consideration.

$$P(i) = e^{-H} \frac{(H)^i}{i!} \qquad\qquad i = 0, 1, 2, \cdots, N-1$$

$$P(N) = 1 - \sum_{i=0}^{N-1} e^{-H} \frac{H^i}{i!}$$

where

$$H = M P_k$$

We shall not specify the relationship of M to N, but one feels that for large M the approximation will be adequate.

In the next section we shall see that reasonable assumptions concerning the attrition of penetrators under attack leads to a stochastic process in which the number of penetrators killed has a truncated Poisson distribution. We shall use cumulative kill potential of defense installations the penetration force encounters as the parameter in this distribution. Moreover, it is our intent to modify kill potential to reflect both defense and offense strategies in achieving their individual goals.

In this section we have seen that the truncated Poisson can be good approximation to the binomial when SLS is the firing doctrine and M is large. Appendix A provides a numerical comparison of the expectation of the truncated Poisson distribution with parameter

$$H = M P_k$$

with the binomials associated with both SLS and MSE firing doctrines when M is not necessarily large.

Section 3. BASIC CONTINUOUS STOCHASTIC PENETRATION MODELS

In this section we consider a number of penetration models that may be solved in closed form by solving an appropriate system of differential equations.[1] In Sections 4 and 5 we modify these models to account for defensive allocation of effort and the effects of penetration aids and tactics. Selecting an appropriate stochastic model is not routine. The selection depends upon an assessment of how the defender will utilize his resources and manage his defense. The basic question is whether a defender utilizes as much of his resources as possible or whether he paces his expenditure of ammunition so that he will be in a position to defend against following attacks.

3.1 A BASIC POISSON ATTRITION MODEL

We assume that N identical penetrators begin a penetration of enemy territory at $x = 0$. In penetrating from 0 to x the penetrating force encounters a number of defensive sites of the same type. We assume there exists a function $\lambda(x)$ having the property that, no matter how many penetrators have been killed over $(0,x)$ and providing at least one penetrator is alive, then the probability that a kill occurs over $(x, x + \Delta x)$ is $\lambda(x)\Delta x + 0(\Delta x)$ where

$$\lim_{\Delta x \to 0} \frac{0(\Delta x)}{\Delta x} = 0 \ .$$

Moreover, the probability of more than one kill over $(x, x + \Delta x)$ is $0(\Delta x)$. If N penetrators have been killed over $(0,x)$ the probability of an additional kill over $(x, x + \Delta x)$ is 0.

These considerations lead to a system of stochastic difference equations for $P_i(x)$, the probability that i penetrators are killed over $(0,x)$. We have

$$P_0(x+\Delta x) = P_0(x)(1-\lambda(x)\Delta x) + 0(\Delta x)$$

$$P_i(x+\Delta x) = P_i(x)(1-\lambda(x)\Delta x) + P_{i-1}(x)\lambda(x)\Delta x + 0(\Delta x)$$
$$i = 1,2,\cdots,N-1$$

$$P_N(x+\Delta x) = P_N(x) + P_{N-1}(x)\lambda(x)\Delta x + 0(\Delta x)$$

---

[1] W. Feller. *An Introduction to Probability Theory and Its Applications*. New York, John Wiley & Sons, Inc., 1967. Chapter 17.

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Dividing each equation by Δx and taking the limit as $\Delta x \to 0$ yields the system of stochastic differential equations,

$$P_0'(x) = -\lambda(x) \; P_0(x)$$

$$P_i'(x) = -\lambda(x) \; P_i(x) + \lambda(x) \; P_{i-1}(x) \qquad\qquad i = 1,2,\cdots,N-1$$

$$P_N'(x) = \lambda(x) \; P_{N-1}(x)$$

If $P_0(0) = 1$, the solution of this system of differential equations is

$$P_i(x) = \frac{[H(x)]^i}{i!} \; e^{-H(x)} \qquad\qquad i = 0,1,\cdots,N-1$$

$$P_N(x) = 1 - \sum_{i=0}^{N-1} P_i(x)$$

where

$$H(x) = \int_0^x \lambda(y)\,dy$$

We define $\lambda(x)$ to be the kill intensity of the defense at x. Based on the results of Section 2 and Appendix A, we assume $H(x)$ gives the cumulative kill potential that will be brought to bear on the penetrating force over $(0,x)$. We note that if $Q_i(x)$ is defined to be the probability that i penetrators are alive at x, then

$$Q_i(x) = P_{N-i}(x) \qquad\qquad i = 0,1,\cdots,N$$

so

$$Q_0(x) = 1 - \sum_{i=1}^{N} Q_i(x)$$

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where

$$Q_i(x) = \frac{[H(x)]^{N-i}}{(N-i)!} e^{-H(x)} \qquad i = 1, 2, \cdots, N$$

Methods of estimating $H(x)$ as a function of defensive resource management policies and penetration force size will be given in Section 4. Degradations to $H(x)$ due to penetration aids are presented in Section 5. For now we turn our attention to a model in which the kill intensity is allowed to be a function of penetrator force size.

3.2 A MODEL WHICH ASSUMES ATTRITION IS PROPORTIONAL TO FORCE SIZE

One reason for allowing kill intensity to depend on force size is the simple fact that a reasonable defensive firing doctrine would allocate more kill intensity to larger forces. After all, the defense must balance available resources to defend over extended periods of time. The assumption that defensive resource allocation will be related to the size of the penetration force is not unreasonable.

In this case it is a little easier to set up the differential equations in terms of the number surviving over $(0,x)$. If i penetrators survive over $(0,x)$, we let the kill intensity at x be

$$\lambda_i(x) = i \, \phi(x)$$

where

$$\phi(x) = \lambda(x)/N^*$$

where $N^* \geq N$ relates to the defense policy of allocating kill intensity to various sizes penetration forces. This guarantees that

$$\lambda_i(x) \leq \lambda(x) \qquad i = 0, 1, 2, \cdots, N$$

and assumes that kill intensity is allocated proportional to the size of the penetration force. By setting $N^* = N$, the kill intensity when none are killed is the same as in 3.1. The differential equations are

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$$Q_0'(x) = \phi(x) \, Q_1(x)$$

$$Q_i'(x) = -i \, \phi(x) \, Q_i(x) + (i + 1) \, \phi(x) \, Q_{i+1}(x)$$

$$i = 1, 2, \cdots, N-1$$

$$Q_N'(x) = N \, \phi(x) \, Q_N(x)$$

with solution

$$Q_i(x) = \binom{N}{i} \left(1 - e^{-\Phi(x)}\right)^{N-i} e^{-i\Phi(x)}$$

where

$$\Phi(x) = \int_0^x \phi(x)\,dx = \int_0^x \lambda(x)/N^* = H(x)/N^*$$

The probability distribution of the number killed is given by

$$P_i(x) = Q_{N-i}(x)$$

$$= \binom{N}{i} \left(1 - e^{-\Phi(x)}\right)^i e^{-(N-i)\Phi(x)}$$

3.3 MIXTURES OF MODELS IN 3.1 AND 3.2

In the initial phase of a penetration it may be appropriate to allocate kill intensity proportional to the force size. In later phases, however, the defense may be willing to allocate considerable kill potential to even small forces. This suggests the possibility of assuming that different models hold over different subintervals of $(0,x)$. Suppose, for example, the model of (3.2) holds over $(0,x_1)$ and the model of (3.1) holds over (x_1,x). In this case,

$$Q_i(x) = \binom{N}{i} \left(1 - e^{-\Phi(x)}\right)^{N-i} e^{-i\Phi(x)} \qquad 0 \le x \le x_1$$

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and

$$Q_i(x) = \sum_{j=i}^{N} Q_j(x_1) \, Q_{i|j}(x_1, \, x) \qquad\qquad x_1 < x$$

where

$$Q_{i|j}(x_1, \, x) = \frac{[H(x_1, \, x)]^{j-i}}{(j - i)!} \, e^{-H(x_1,x)}$$

$$Q_{0|j}(x_1, \, x) = 1 - \sum_{i=1}^{j} Q_{i|j}(x_1, \, x)$$

and

$$H(x_1, \, x) = \int_{x_1}^{x} \lambda(y) \, dy$$

More generally, suppose one has obtained $Q_i(x)$ over the subintervals $(0, \, x_1)$, $(x_1, \, x_2)$, $\cdots$, $(x_{k-1}, \, x_k)$ and now wants to obtain $Q_i(x)$ for $x > x_k$. In this case,

$$Q_i(x) = \sum_{j=i}^{N} Q_j(x_k) \, Q_{i|j}(x_k, \, x) \qquad\qquad x_k < x$$

where $Q_j(x_k)$ is known and $Q_{i|j}(x_k, \, x)$ depends upon the model assumed to hold over $x > x_k$. If, for $x > x_k$, the model of 3.1 holds

$$Q_{i|j}(x_k, \, x) = \frac{[H(x_k, \, x)]^{j-i}}{(j - i)!} \, e^{-H(x_k,x)} \qquad\qquad i = 1,2,\cdots,j$$

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$$Q_{0|j}(x_k, x) = 1 - \sum_{i=1}^{j} Q_{i|j}(x_k, x)$$

$$H(x_k, x) = \int_{x_k}^{x} \lambda(y) \, dy$$

If, for $x > x_k$, the model of 3.2 holds

$$Q_{i|j}(x_k, x) = \binom{j}{i}\left(1 - e^{-\Phi(x_k,x)}\right)^{j-i} e^{-i\Phi(x_k,x)}$$

$$i = 0,1,\cdots,j$$

$$\Phi(x_k, x) = \frac{1}{N^*_{k+1}} \int_{x_k}^{x} \lambda(y) \, dy$$

where $N^*_{k+1} \geq j$ is a defense scaling parameter for (x_k, x) when j are alive at x_k.

3.4 A COMBINED MODEL

Another method of combining the two models assumes that the kill intensity at x when i penetrators are alive is

$$\lambda_i(x) = \omega(x) + i\phi(x) \qquad\qquad i = 1,2,\cdots,N$$

$$\lambda_0(x) = 0$$

with

$$\omega(x) + N*\phi(x) = \lambda(x)$$

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If $\omega(x)$ is specified,

$$\phi(x) = \frac{\lambda(x) - \omega(x)}{N*}$$

If $\phi(x)$ is specified,

$$\omega(x) = \lambda(x) - N*\phi(x)$$

For general $\omega(x)$ and $\phi(x)$, this problem can not be solved easily in closed form except for small N.

The differential equations are

$$P_i'(x) = -(\omega(x)+(N-i)\phi(x))P_i(x) + (\omega(x)+(N-i+1)\phi(x))P_{i-1}(x)$$

$$P_N'(x) = -(\omega(x)+\phi(x))P_{N-1}(x)$$

We have

$$P_0(x) = e^{-\Omega(x)-N\Phi(x)}$$

where

$$\Omega(x) = \int_0^x \omega(y)\,dy$$

and

$$P_1(x) = \left[\int_0^x \omega(y)e^{-\Phi(y)}\,dy + N\left[1-e^{-\Phi(x)}\right]\right]e^{-[\Omega(x)+(N-1)\Phi(x)]}$$

One may proceed solving the linear differential equations in turn to obtain P_2, P_3, etc. These expressions get quite complex very quickly and it may actually be better to solve these equations numerically.

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3.4.1 A Special Case

In the special case that $\omega(x) = \omega$ and $\phi(x) = \phi$ independent of x the system of differential equations above may be solved in closed form. One may verify that the solution is given by

$$P_0(x) = e^{-(\omega x + N\phi x)}$$

$$P_i(x) = e^{-(\omega x + (N-i)\phi x)}\left(1 - e^{-\phi x}\right)^i \prod_{k=0}^{i-1} \frac{(\omega + (N-k)\phi)}{i!\phi^i}$$

$$i = 1, 2, \cdots, N-1$$

$$P_N(x) = 1 - \sum_{i=0}^{N-1} P_i(x)$$

This closed form solution, together with the idea of combining models that are different for different subintervals of $(0,x)$, provides a reasonable method of solving this problem in general. We merely approximate $\omega(x)$ and $\phi(x)$ by piecewise constant functions with

$$\omega(x) = \omega_k \qquad x_{k-1} \leq x \leq x_k \qquad k = 1, 2, \cdots, m$$

$$\phi(x) = \phi_k \qquad x_{k-1} \leq x \leq x_k \qquad k = 1, 2, \cdots, m$$

where $x_0 = 0$.

Designating the above probabilities by $P_i(x; \omega, \phi, N)$, we have

$$P_i(x) = \sum_{j=0}^{i} P_j(x_{k-1}) P_{i-j}(x - x_{k-1}; \omega_k, \phi_k, N-j) \qquad x_{k-1} \leq x \leq x_k$$

for $k = 1, 2, \cdots, m$ where $P_0(0) = 1$ and

$$P_i(x) = P_i(x; \omega_1, \phi_1, N) \qquad\qquad 0 \leq x \leq x_1$$

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These formulae are easily modified to account for reliability. Suppose each penetrator is subject to a constant failure rate $\beta = \mu v$, where v is the penetration speed. All one need do is replace ϕ_k by $\phi_k + \mu$ in the above formulae. If the penetration force travels a distance a before beginning the penetration at $x = 0$ then

$$P_i(x) = \sum_{j=0}^{i} P_j(x_{k-1}) P_{i-j}(x - x_{k-1}; \omega_k, \phi_k + \mu, N-j)$$

$$x_{k-1} \leq x \leq x_k$$

for $k = 1, 2, \cdots, m$ where

$$P_i(0) = P_i(a; 0, \mu, N) \qquad\qquad i = 0, 1, 2, \cdots, N$$

$$= \binom{N}{i} \left(1 - e^{-\mu a}\right)^i e^{-(N-i)\mu a}$$

Another method of obtaining an approximate solution is to obtain the average of ω and ϕ over $(0, x)$.

$$\omega = \int_0^x \omega(y)\, dy / x = \Omega(x)/x$$

and

$$\phi = \Phi(x)/x$$

giving

$$P_0(x) = e^{-(\Omega(x) + N\Phi(x))}$$

$$P_i(x) = e^{-(\Omega(x) + (N-i)\Phi(x))} \left(1 - e^{-\Phi(x)}\right)^i \prod_{k=0}^{i-1} \frac{(\Omega(x) + (N-k)\Phi(x))}{i!\ \Phi(x)^i}$$

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$$P_N(x) = 1 - \sum_{i=0}^{N-1} P_i(x)$$

3.4.2 The Expected Value of Number Killed

Realizing that

$$\frac{dE(x)}{dx} = \sum_{i=1}^{N} iP_i'(x)$$

where $E(x)$ is the expected number of penetrators killed over $(0,x)$, we obtain, after some algebraic manipulation of the differential equations in this section,

$$\frac{dE(x)}{dx} + \phi(x)E(x) = \omega(x)(1-P_N(x)) + N\phi(x)$$

with solution

$$E(x) = e^{-\Phi(x)} \int_0^x \omega(y)(1-P_N(y))e^{\Phi(y)}dy$$

$$+ N\left[1-e^{-\Phi(x)}\right]$$

In most instances $P_N(y) \cong 0$ for all y, so

$$E(x) \cong e^{-\Phi(x)} \int_0^x \omega(y)e^{\Phi(y)}dy + N\left[1-e^{-\Phi(x)}\right]$$

Thus, if one is only interested in the expected number killed, a reasonable estimate of this expectation is available.

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3.5 NUMERICAL SOLUTION OF DIFFERENTIAL-DIFFERENCE EQUATIONS

We assume that the kill intensity at x given that i penetrators are alive is $\lambda_i(x)$. We have considered the case where

$$\lambda_i(x) = \omega(x) + i(\phi(x)+\mu)$$

and μ is a reliability factor. We now allow $\lambda_i(x)$ to be quite general. The basic difference equations are

$$Q_i(x+\Delta x) = (1-\lambda_i(x)\Delta x)Q_i(x) + \lambda_{i+1}(x)\Delta x Q_{i+1}(x)$$

$$i = 0,1,\cdots,N$$

where $\lambda_0(x) = 0$ and $Q_{N+1}(x) = 0$ for all x and

$$Q_i(0) = \binom{N}{i}e^{-\mu ai}(1-e^{-\mu a})^{N-i}$$

with the understanding that

$$Q_N(0) = 1$$

if $\mu = 0$ or $a = 0$.

For a fixed Δx the computation is straightforward. It is clear that both accuracy and cost will depend on Δx and experience with the solution should dictate a choice of Δx which balances accuracy with cost.

3.6 PENETRATORS OF DIFFERENT TYPES

If the penetration force consists of J different types of penetrators, then the stochastic difference equations are

$$Q_{i_1,i_2,\cdots,i_J}(x+\Delta x) = \left(1 - \sum_{j=1}^{J}\lambda_j(x;i_1,i_2,\cdots,i_J)\Delta x\right)Q_{i_1,\cdots,i_J}(x)$$

$$+ \sum_{j=1}^{J}\lambda_j(x;i_1,\cdots,i_{j+1},\cdots,i_J)\Delta x Q_{i_1,\cdots,i_{j+1},\cdots,i_J}(x)$$

where $\lambda_j(x; i_1, \cdots, i_J)$ is the kill intensity against the jth type of penetrator at x when $i_j, j=1, \cdots, J$ penetrators of type j are alive.

The differential equations become

$$Q'_{i_1, \cdots, i_J}(x) = -\sum_{j=1}^{J} \lambda_j(x; i_1, i_2, \cdots, i_J) Q_{i_1, i_2, \cdots, i_J}(x)$$

$$+ \sum_{j=1}^{J} \lambda_j(x; i_1, i_2, \cdots, i_{j+1}, \cdots, i_J) Q_{i_1, i_2, \cdots, i_{j+1}, \cdots, i_J}(x)$$

If

$$\lambda_j(x; i_1, \cdots, i_j, \cdots, i_J) = \lambda_{i_j}(x)$$

independent of $i_1, \cdots, i_{j-1}, i_{j+1}, \cdots, i_J$ for all j, then

$$Q_{i_1, i_2, \cdots, i_J}(x) = \prod_{j=1}^{J} Q_{i_j}(x)$$

where $Q_{i_j}(x)$ satisfies

$$Q'_{i_j}(x) = -\lambda_{i_j}(x) Q_{i_j}(x)$$

$$+ \lambda_{i_{j+1}} Q_{i_{j+1}}(x)$$

To verify that this is the solution we have from the assumed form of $Q_{i_1,\cdots,i_J}(x)$

$$Q'_{i_1,\cdots,i_J}(x) = \sum_{j=1}^{J} \prod_{m\neq j}^{J} Q_{i_m}(x) Q'_{i_j}(x)$$

$$= - \sum_{j=1}^{J} \prod_{m\neq j}^{J} Q_{i_m}(x) \left[\lambda_{i_j}(x) Q_{i_j}(x) \right]$$

$$+ \sum_{j=1}^{J} \prod_{m\neq j}^{J} Q_{i_m}(x) \left[\lambda_{i_{j+1}}(x) Q_{i_{j+1}}(x) \right]$$

$$= - \sum_{j=1}^{J} \lambda_{i_j}(x) Q_{i_1,\cdots,i_J}(x)$$

$$+ \sum_{j=1}^{J} \lambda_{i_{j+1}}(x) Q_{i_1,\cdots,i_{j+1},\cdots,i_J}(x)$$

This is an important result, for it says that survival of different types of penetrators in a penetration group may be treated separately providing the attrition rate of a given type does not depend on the number of penetrators of other types that are alive. As one might expect, this is rarely the case when penetration aids and uncertainties in the classification of target types are incorporated in the modeling.

We may certainly extend the numerical solution suggested in 3.5 to the case where there are a number of penetrator types and independence does not hold. Our concern is over sheer magnitude of the number of

calculations required. Suppose, for example, we have a penetration
force of five types with nine penetrators per type. In the case of
independence, 50 probabilities must be obtained each time the joint
probability distribution is determined at $x + \Delta x$ from the results at x.
For non-independence, the number 50 grows to 100,000 indicating the
computational difficulties encountered when independence does not hold.

By combining expected value concepts with the stochastic differential
equations that arise when independence does not hold, one may solve a
sequence of independent problems that approximates the dependent case.
We address this problem in Sections 4 and 5.

Section 4. CUMULATIVE KILL POTENTIAL FROM THE
DEFENSE POINT OF VIEW

In this section we want to develop some simple models for estimating cumulative kill potential, $H(x)$, where $H'(x) = \lambda(x)$ is a basic input to the models discussed in Section 3. An input to this calculation is the kill potential of an individual site and considerations of how that site will utilize its available kill potential. Cumulative kill potential calculations are extended to include different types of defense units and the problem of allocation of kill potential to penetrators of different types is also discussed. These considerations generally lead to penetration models which may be solved in closed form using the models of Section 3.

There are aspects of the defense problem that lead to penetration models that must be solved numerically or involve dependence considerations. One problem of special importance is the effect of mistakes in classification of penetrator types. Modeling considerations that allow one to simplify this problem are discussed.

4.1 KILL POTENTIAL OF A SINGLE SITE

We consider a group of penetrators at altitude h and offset from the defense site a distance y. The defense can get off M shots with P_k's as indicated in Figure 1.

FIGURE 1.

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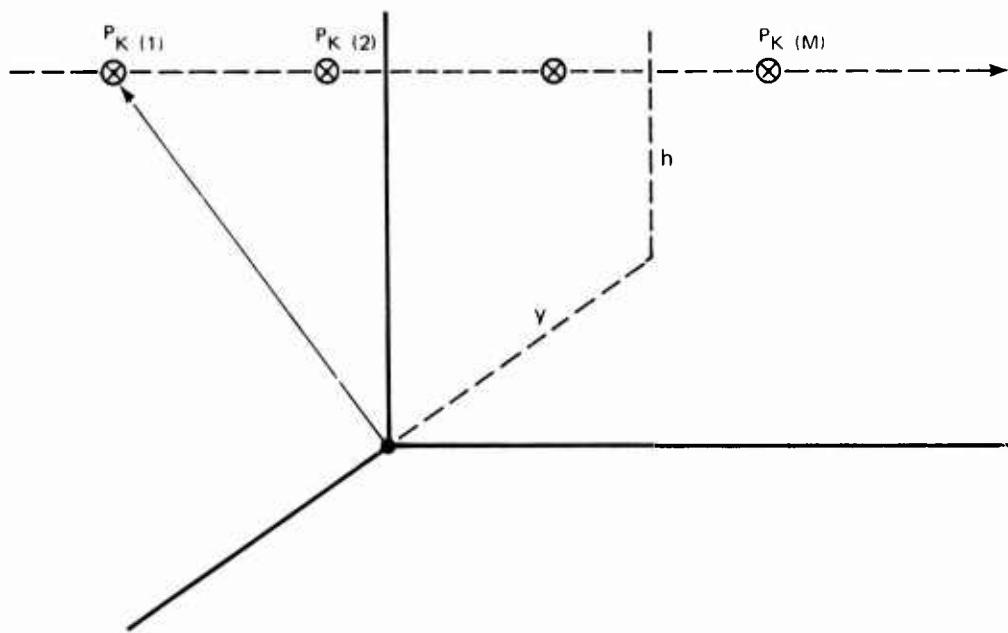
Actually M and the P_k's depend on h, y, and v, the speed of the penetrators. As a consequence, the kill potential of the site is

$$K_p(h,v,y) = \sum_{m=1}^{M(h,v,y)} P_k(h,v,y,m)$$

where $M(h,v,y)$ is the number of shots as a function of h, v, and y and $P_k(h,v,y,m)$ is the kill probability of the mth shot as a function of h, v, and y. In general, y is considered uniformly distributed over $(0, R(h))$ where $R(h)$ is the effective range of the site as a function of h. Consequently, we set the kill potential of the site equal to

$$K_p(h,v) = \frac{1}{R(h)} \int_0^{R(h)} K_p(h,v,y)\,dy$$

Another consideration of importance to the defense is expected weapons usage

$$W(h,v) = \frac{1}{R(h)} \int_0^{R(h)} M(h,v,y)\,dy \equiv M(h,v)$$

The average probability of kill at offset y is

$$P_k(h,v,y) = \sum_{m=1}^{M(h,v,y)} P_k(h,v,y,m)/M(h,v,y)$$

so

$$K_p(h,v) = \frac{1}{R(h)} \int_0^{R(h)} M(h,v,y)P_k(h,v,y)\,dy$$

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and we define $\bar{P}_k(h,v)$ so that

$$K_p(h,v) = M(h,v)\bar{P}_k(h,v) = W(h,v)\bar{P}_k(h,v)$$

4.1.1 The Effect of Salvo Firing

Should the defense elect to fire in salvos with r missiles per salvo then

$$P_k(h,v,y;r) = \sum_{m=1}^{M(h,v,y)} (1-(1-P_k(h,v,y,m))^r)\big/M(h,v,y)$$

and

$$K_p(h,v;r) = \frac{1}{R(h)} \int_0^{R(h)} M(h,v,y)P_k(h,v,y;r)\,dy$$

and expected weapons usage is

$$W(h,v;r) = rM(h,v)$$

Again we define $\bar{P}_k(h,v;r)$ so

$$K_p(h,v,r) = M(h,v)\bar{P}_k(h,v;r) = W(h,v;r)\,\frac{\bar{P}_k(h,v;r)}{r}$$

4.1.2 Selective Salvo Firing

We consider the permutation $i_1,i_2,\cdots,i_{M(h,v,y)}$ of integers $1,2,\cdots,M(h,v,y)$ with the property that

$$P_k(h,v,y,i_1) \geq P_k(h,v,y,i_2),\cdots, \geq P_k(h,v,y,i_{M(h,v,y)})$$

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Next we define

$$I_{\hat{M}} = \{i_j \; : \; j = 1,2,\cdots,\hat{M}\}$$

so that $I_{\hat{M}}$ indexes the integers associated with the $\hat{M}$ largest P_k's, where $\hat{M}(h,v,y) \leq M(h,v,y)$

$$I_{\hat{P}_k} = \{i_j \; : \; P_k(h,v,y,i_j) \geq \hat{P}_k\}$$

where this set may be empty. Thus $I_{\hat{P}_k}$ identifies those integers such that P_k's exceed some preselected value $\hat{P}_k$. We let $\hat{N}(h,v,y)$ be the number of integers in $I_{\hat{P}_k}$.

A reasonable defense policy is to set $\hat{M}$, $\hat{P}_k$ and r. Now

$$P_k^*(h,v,y;r) = \sum_{m=1}^{M^*(h,v,y)} (1-(1-P_k(h,v,y,i_m))^r)/M^*(h,v,y)$$

where

$$M^*(h,v,y) = \min(\hat{M}(h,v,y),\hat{N}(h,v,y))$$

so

$$K_p^*(h,v;r) = \frac{1}{R(h)} \int_0^{R(h)} M^*(h,v,y)P_k^*(h,v,y;r)dy$$

We let

$$M^*(h,v) = \frac{1}{R(h)} \int_0^{R(h)} M^*(h,v,y)dy$$

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and define $\bar{P}_k^*(h,v;r)$, so

$$P_k^*(h,v,r) = M^*(h,v)\bar{P}_k^*(h,v;r) = W^*(h,v;r)\,\frac{\bar{P}_k(h,v;r)}{r}$$

where

$$W^*(h,v;r) = rM^*(h,v)$$

4.1.3 The Effect of Penetration Force Size

The above calculations treat the penetration force as a point which moves through the region of defense effectiveness. For a small number of tightly grouped penetrators this may be adequate, but for larger penetration forces practicing jinking and maneuvering and trying to control decoys the size of the force may be an important consideration. Generally speaking, a larger penetrating force will increase the firing opportunities available to the defense and perhaps allow him to select opportunities with higher P_k's.

The easiest way to account for this effect without complicating the model is to assume the penetration force of size N occupies a certain volume in space which does not change as penetrators are killed. Thus, this fixed volume implies that $M(h,v,y)$ and $P_k(h,v,y,m)$ may be determined as functions of N, the size of the penetrating force. Regrouping to present a smaller volume as penetrators are killed would make M and P_k depend on the number alive at any instant of time. This complicates the modeling considerably and will not be considered in this report.

4.1.4 Interceptor Aircraft

If the defense site is an interceptor aircraft, then kill potential may be computed in much the same way. We set

$$K_p(h,v) = M(h,v)P_k(h,v)$$

for aircraft where $M(h,v)$ is the number of passes and $P_k(h,v)$ is the kill probability of a pass. We allow both of these quantities to depend on altitude and speed of the penetration force. As before, we can also make these quantities dependent on N, which may increase M and perhaps P_k.

4.2 CUMULATIVE KILL POTENTIAL

Since kill potential over a number of defense sites is additive, it only remains to determine the number of sites encountered in a penetration of depth x. We let $f(y)$ be the density of defense sites at penetration position y, then the number of sites encountered is

$$N(x) = 2R(h) \int_0^x f(y)\,dy$$

and the cumulative kill potential is

$$H(x) = 2R(h)K_p(h,v) \int_0^x f(y)\,dy$$

If we allow h and v to depend upon penetrator force location,

$$H(x) = 2 \int_0^x R(h(y))K_p(h(y),v(y))f(y)\,dy$$

which implies

$$\lambda(x) = 2R(h(x))K_p(h(x),v(x))f(x)$$

4.2.1 Nonhomogeneous Defense

Cumulative kill potential is easily extended for a mixture of different defense types. Once again, since kill potential adds, we merely index the above results for defense type k. Thus

$$H_k(x) = 2 \int_0^x R_k(h(y))K_{pk}(h(y),v(y))f_k(y)\,dy$$

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with

$$\lambda_k(x) = 2R_k(h(x))K_{pk}(h(x),v(x))f_k(x)$$

The total cumulative kill potential is

$$H(x) = \sum_{k=1}^{K} H_k(x)$$

which implies that

$$\lambda(x) = \sum_{k=1}^{K} \lambda_k(x)$$

4.3 ALLOCATION OF DEFENSE CAPABILITY TO PENETRATORS OF DIFFERENT TYPES

Penetrators of different types may have markedly different penetration capabilities. This suggests making the above results dependent on penetrator type. Thus, for penetrator type j, we have

$$\lambda_{kj}(x) = 2R_{kj}(h(x))K_{pkj}(h(x),v(x))f_k(x)$$

as the kill intensity of type k defenses against type j penetrators. With

$$H_{kj}(x) = \int_{0}^{x} \lambda_{kj}(y)\,dy$$

as the cumulative kill potential if type k defenses are used against type j penetrators. The total cumulative kill potential that may be brought to bear against type j penetrators is

$$H_{\cdot j}(x) = \sum_{k=1}^{K} H_{kj}(x)$$

The reader should compare $H(x)$ in 4.2 and $H_{\cdot j}(x)$ and realize that $H(x)$ is $H_{\cdot j}(x)$ when one type of penetrator is considered.

4.3.1 Fixed Allocation of Defense for a Nonhomogeneous Penetration Force

We want to allow the penetration force to be made up of penetrators of different types. For the moment let us assume that the defense fixes its allocation of defense to penetrators as a function of penetrator force location. We let $\alpha_{kj}(x)$ be the fraction of defense capability of type k allocated to penetrators of type j at penetration location x, where

$$0 \le \alpha_{kj}(x) \le 1 \qquad\qquad \text{all k and j}$$

$$\sum_{j=1}^{J} \alpha_{kj}(x) \le 1 \qquad\qquad \text{all k}$$

Now

$$H_{kj\alpha}(x) = \int_{0}^{x} \lambda_{kj}(y)\alpha_{kj}(y)\,dy$$

and

$$H_{\cdot j\alpha}(x) = \sum_{k} \int_{0}^{x} \lambda_{kj}(y)\alpha_{kj}(y)\,dy$$

is the cumulative kill potential allocated to penetrators of type j. The kill intensity is

$$\lambda_{\cdot j\alpha}(x) = \sum_{k=1}^{K} \lambda_{kj}(x)\alpha_{kj}(x)$$

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We note that these results still allow one to use the models in Section 3 which require, at most, the numerical integration of $\lambda_{.j\alpha}(x)$ or $\omega_{.j\alpha}$ and $\phi_{.j\alpha}$ functions related to it. That is, numerical integration of the basic differential equations and problems of dependence among penetrator types have still been avoided.

4.3.2 Classification of Penetrator Type and Adaptive Allocation of Defense

If one can distinguish penetrators of different types, then $\alpha_{kj}(x)$ may be specified. Classification may not be perfect; moreover, $\alpha_{kj}(x)$ should be 0 if no penetrators of type j are alive.

We now consider an allocation process which takes these factors into account. We assume a penetrator of type j is classified as type r with probability $P_{jr}(x)$. If a penetrator of type r has relative weight $W_r(x)$ and if i_j penetrators of type j are alive at x, then

$$\alpha_{kj}(x) = C_k i_j \sum_r P_{jr}(x)W_r(x)$$

$$= C_k i_j W_j^*(x)$$

where

$$W_j^*(x) = \sum_r P_{jr}(x)W_r(x)$$

and C_k is chosen, so

$$\sum_j \alpha_{kj}(x) = \theta_k(x) \leq 1$$

where $\theta_k(x)$ is a defense decision function relating to the use of k type weapons. Thus,

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$$\alpha_{kj}(x) = \frac{i_j W_j^*(x)\theta_k(x)}{\sum_j i_j W_j^*(x)}$$

$$\alpha_{kj}(x) = C_k(x; i_1, i_2, \cdots, i_J) i_j W_j^*(x)$$

where

$$C_k(x; i_1, \cdots, i_J) = \left[\sum_j i_j W_j^*(x) \right]^{-1} \theta_k(x)$$

In this section we find ourselves in the dependent case with

$$\lambda_j(x; i_1, i_2, \cdots, i_J) = i_j W_j^*(x) \sum_{k=1}^{K} \lambda_{kj}(x) C_k(x; i_1, \cdots, i_J)$$

Also, one is immediately involved in the numerical solution of stochastic-differential equations.

4.3.3 Adaptive Allocation of Defense With Independence of Penetrator Type

We have seen that adaptive allocation of defense leads to dependence among types of penetrator groups. Moreover, to carry out the solution would require the numerical solution of the appropriate stochastic-differential equations. The difficulty generated in 4.3.2 is similar to complexities that arise in the analysis of penetration aid effects. We now consider several methods of handling this problem.

The first step comes in analyzing where the difficulty occurs. We have

$$\lambda_j(x; i_1, \cdots, i_J) = \sum_{k=1}^{K} \alpha_{kj}(x) \lambda_{kj}(x)$$

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where

$$\lambda_{kj}(x) = 2R_{kj}(h(x))K_{pkj}(h(x),v(x))f_k(x)$$

$$\alpha_{kj}(x) = \frac{i_j W_j^*(x)\theta_k(x)}{\sum\limits_{j} i_j W_j^*(x)}$$

The problem is not in the kill intensity portion of the calculation, but in the allocation portion. The problem is really one of approximating $\alpha_{kj}(x)$.

If we can assume that, for all j, there is an attrition function $a(x)$, independent of j so that

$$i_j(x) = N_j a(x)$$

then

$$\alpha_{kj}(x) = \frac{N_j W_j^*(x)\theta_k(x)}{\sum\limits_{j} N_j W_j^*(x)}$$

Now $\lambda_j(x;i_1,\cdots,i_J) = \lambda_j(x)$, so independence holds. In fact, the model of 3.1 or 3.2 may be applied to yield $Q_{i_j}(x)$ for each j independently.

One might be tempted to set

$$\alpha_{kj}(x) = \frac{i_j W_j^*(x)\theta_k(x)}{\sum\limits_{j} N_j W_j^*(x)}$$

and then use the model of 3.2. This is certainly possible, but the intent of model 3.2 is to consider those cases where $\lambda_j(x)$ may be reduced at the discretion of the defense when penetration force size is small. Another problem is that the above expression summed on j does not give $\theta_k(x)$.

Another possibility is to replace $\alpha_{kj}(x)$ by its expected value

$$E(\alpha_{kj}(x)) = \sum_{i_1,\cdots,i_J} \alpha_{kj}(x)Q_{i_1,\cdots,i_J}(x)$$

Even if $Q_{i_1,\cdots,i_J}(x)$ were known, this procedure would suffer from the fact that, in general,

$$\sum_j E(\alpha_{kj}(x)) \neq \theta_k(x)$$

If, however, we define $E_j(x)$ as the expected number of survivors of type j alive at x and set

$$\bar{\alpha}_{kj}(x) = \frac{E_j(x)W_j^*(x)\theta_k(x)}{\sum_j E_j(x)W_j^*(x)}$$

we have

$$\sum_j \bar{\alpha}_{kj}(x) = \theta_k(x)$$

The only problem remaining is that $E_j(x)$ is not known.

We turn to this problem under more general conditions in 5.1, but for now consider setting

$$\bar{\alpha}_{kj}^{(0)}(x) = \frac{N_j W_j^*(x)\theta_k(x)}{N_j W_j^*(x)}$$

Using the model of 3.1, 3.2, or 3.4, whichever is appropriate, we obtain $Q_{i_j}^{(1)}(x)$ and compute

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$$E_j^{(1)}(x) = \sum_{i_j=0}^{N_j} i_j Q_{i_j}^{(1)}(x)$$

Now

$$\bar{\alpha}_{kj}^{(1)}(x) = \frac{E_j^{(1)}(x) W_j^*(x) \theta_k(x)}{\sum E_j^{(1)}(x) W_j^*(x)}$$

which allows one to obtain $Q_{i_j}^{(2)}(x)$ and hence $E_j^{(2)}(x)$. This procedure is repeated until changes in the $E_j(x)$ calculation are acceptable. This procedure will be expanded upon in Section 5.1.

Section 5. PENETRATION AIDS AND TACTICS

We now consider the effect of various penetration aids and tactics on the kill potential the defense can mount. We shall consider trajectory variations, decoys, ECM, defense suppression, self-defense, and standoff tactics.

The reader should recall that, with reasonable assumptions, we have been able to maintain independence of penetration types. As pointed out in Section 3, if one is forced to go to dependence, then the computing may become excessive for large penetrations. In this section it is virtually impossible to avoid dependence because of the synergistic effects of one type of penetration group with another. Indeed these effects are an important part of penetration aid analysis. Before treating the penetration aid problem we consider a general approximation method which maintains independence while accounting for synergistic effects in an average value way.

5.1 RECURSIVE SOLUTION PROCEDURE TO MAINTAIN INDEPENDENCE AND ALLOW ASSESSMENT OF SYNERGISTIC EFFECTS

Penetration aid analysis will lead to λ_j which depend not just on i_j but also on $i_1, i_2, \cdots$, and i_J. We recall, however, that cumulative expected kill potential has always been the parameter of interest to us. Thus, taking $\lambda_j(x; i_1, \cdots, i_J)$ and replacing it by

$$E[\lambda_j(x)] = \sum_{i_1, \cdots, i_J} \lambda_j(x; i_1, \cdots, i_J) Q_{i_1, \ldots, i_J}(x)$$

presents no conceptual difficulty. The problem is that $Q_{i_1, \ldots, i_J}(x)$ is unknown.

Assuming that one has an initial estimate $E[\lambda_j^{(0)}(x)]$ of $E[\lambda_j(x)]$ then by using a model of 3.1, 3.2, or 3.4 one may obtain $Q_{i_j}^{(0)}(x)$, since independence holds. In general, at the kth iteration one obtains

$$E\left[\lambda_j^{(k)}(x)\right] = \sum_{i_1, \cdots, i_J} \lambda_j(x; i_1, \cdots, i_J) Q_{i_1}^{(k)}(x), \cdots, Q_{i_J}^{(k)}(x)$$

from which $Q_{i_j}^{(k+1)}(x)$ is obtained using a model of 3.1, 3.2, or 3.4.

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This iterative procedure is continued until

$$\max_{x,j} \left| E\left[\lambda_j^{(k)}(x)\right] - E\left[\lambda_j^{(k+1)}(x)\right] \right| < \varepsilon$$

where ε is a small positive number.

Another possibility is to approximate λ_j by $\bar{\lambda}_j(x; E_1(x), \cdots, E_J(x))$ where $E_j(x)$ is the expected number of survivors of type j alive at x. In this case we begin with an approximate solution $E_j^{(0)}(x)$ which yields $Q_{i_j}^{(1)}(x)$. At the kth iteration, $\bar{\lambda}_j(x; E_1^{(k)}(x), \cdots, E_J^{(k)}(x))$ yields $Q_{i_j}^{(k+1)}(x)$, using a model of 3.1, 3.2, or 3.4. Now

$$E_j^{(k+1)}(x) = \sum i_j Q_{i_j}^{(k+1)}(x)$$

This iterative procedure stops when

$$\max_{j,x} \left| E_j^{(k)}(x) - E_j^{(k+1)}(x) \right| < \varepsilon \quad .$$

In this report we shall concentrate on this latter approach. We have already seen in 4.3.3 that it produces less complexity than dealing with $E(\lambda_j(x))$. It still satisfies the requirement of allowing for the evaluation of synergistic effects among penetration aids.

5.2 TRAJECTORY VARIATION

The model already allows altitude h and speed v to depend on penetrator location x. If one has available a model for the effects of jinking and/or terrain following, inputs to this model in terms of M and P_k for a single defense site may be modified accordingly. If M and/or P_k are not reduced sufficiently, then the effect of maneuvering may not reduce the kill potential of the site below the level the defense intends to expend in the first place. In addition, β, the reliability failure rate, will increase under a maneuvering tactic and the average speed of penetration will decrease, further increasing $\mu = \beta/v$.

Thus the model as structured allows for jinking and terrain-following inputs from other sources and allows for an evaluation of these tactics through their effects on M, P_k, and μ, the reliability factor.

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5.3 DECOYS

The model easily handles decoys by allowing one of the penetrating subgroups to be a group of decoys. Important parameters in the model with regard to the decoys subgroup j are μ_j, the reliability factor; $W_j(x)$, the weight the defense places on decoys; and $P_{jk}(x)$, the probability that a decoy is classified as a penetrator of type k. It is not at all obvious that the defense should place a weight of 0 on decoys. In fact, the evaluation of defense policy with regard to assignment of all weights is an important attribute of the model.

5.4 ELECTRONIC COUNTERMEASURES

Electronic countermeasures (ECM) may affect defense kill potential in several ways. The user may specify any or all of the following possibilities.

5.4.1 Noise Jamming

If there are i_j penetrators of type j alive at x, P(j) is the jamming power of a penetrator of type j, and A(j) is the average radar reflective area of a type j penetrator, then we define expected noise jamming power of the group at x to be

$$\bar{P}(x) = \sum_{j=1}^{J} E_j(x)P(j)$$

With simplifying assumptions regarding penetrator volume and resolvability of the defense radar, the expected radar reflective area of the group at x is

$$\bar{A}(x) = \sum_{j=1}^{J} E_j(x)A(j)$$

where $E_j(x)$ is the expected number of penetrators of type j alive at x.

Essentially noise jamming reduces the range at which the defense can take action. If $R_k(x)$ is the burnthrough range at x for defense type k, then we estimate[2] $R_k(x)$ by

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[2] *Naval Operations Analysis*. U. S. Naval Institute, Annapolis, Md., 1968. (Library of Congress Card 67-28645.)

$$R_k(\vec{E}(x)) = \sqrt{\frac{P_T(k)G(k)\bar{A}(x)}{4\pi\bar{P}(x)}}$$

where $P_T(k)$ is the transmitted power associated with a defense of type k, $G(k)$ is the associated antenna gain, and

$$\vec{E}(x) = [E_1(x),\cdots,E_J(x)]$$

Thus the effective range of a defense of type k against a penetrator of type j is

$$R_{kj}(h(x),\vec{E}) = \max\left[0,\min[R_k(\vec{E}(x)) - v\Delta T_{kj},R_{kj}(h(x))]\right]$$

where ΔT_{kj} is the delay for the system to acquire once burnthrough has occurred. Now in computing kill intensity one uses $R_{kj}(h(x),\vec{E})$ in place of $R_{kj}(h(x))$. If one can reasonably assume that jamming reduces the basic single-shot P_k's, this effect can also be considered.

Thus with noise jamming

$$\lambda_{kj}(x,\vec{E}) = 2R_{kj}(h(x),\vec{E})K_{pkj}(h(x),v(x),\vec{E})f_k(x)$$

where

$$K_{pkj}(h(x),v(x),\vec{E}) = \frac{1}{R_{kj}(h(x),\vec{E})} \int_0^{R_{kj}(h(x),\vec{E})} M_{kj}(h(x),v(x),\vec{E},y)P_{Kkj}(h(x),v(x),\vec{E},y)dy$$

and the expressions under the integral sign are obtained using $R_{kj}(h(x),\vec{E})$. We also have $K_{pkj}(h(x),v(x),\vec{E}) = M_{kj}(h(x),v(x),\vec{E})\bar{P}_{Kkj}(h(x),v(x),\vec{E})$ where

$$M_{kj}(h(x),v(x),\vec{E}) = \frac{1}{R_{kj}(h(x),\vec{E})} \int_0^{R_{kj}(h(x),\vec{E})} M_{kj}(h(x),v(x),\vec{E},y)dy$$

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and $\bar{P}_{Kkj}(h(x),v(x),\vec{E})$ is determined accordingly.

Even though there is an obvious cancellation of R_{kj} terms, it is instructive to maintain the form

$$\lambda_{kj}(x,\vec{E}) = 2R_{kj}(h(x),\vec{E})M_{kj}(h(x),v(x),\vec{E})\bar{P}_{Kkj}(h(x),v(x),\vec{E})f_k(x)$$

and note that noise jamming essentially degrades the range of effectiveness of a site, thus changing M and P_K for a single site and reducing the number of sites encountered.

5.4.2 Deception

One important aspect of ECM is to make the classification problem of the defense more difficult. If a particular class, j^*, is devoted to providing an ECM capability, then we may allow the classification probability $P_{jr}(x)$ to depend on i_{j^*}, the number in this subgroup. Referring to Section 4.3.3, we have

$$\alpha_{kj}(x) = \frac{i_j W_j^*(x;i_{j^*})\theta_k(x)}{\sum i_j W_j^*(x;i_{j^*})}$$

where

$$W_j^*(x;i_{j^*}) = \sum_r P_{jr}(x;i_{j^*})W_r(x)$$

Proceeding as in 4.3.3, we have

$$\bar{\alpha}_{kj}(x;E_{j^*}) = \frac{E_j(x)W_j^*(x;E_{j^*}(x))\theta_k(x)}{\sum_{j=1}^{J} E_j(x)W_j^*(;E_{j^*}(x))}$$

where

$$W_j^*(x;E_{j^*}(x)) = \sum_r P_{jr}(x;E_{j^*}(x))W_r(x)$$

Now we define

$$\lambda_{kj\alpha}(x,\vec{E},E_{j*}) = \bar{\alpha}_{kj}(x;E_{j*})\lambda_{kj}(x,\vec{E})$$

yielding kill intensity due to jamming, deception, and defensive allocation.

There will be an attempt throughout the remainder of the report to make the notation as descriptive as possible of the effects that are being considered. Thus

$$\lambda_{kj\alpha}(x,\vec{E}) = \bar{\alpha}_{kj}(x)\lambda_{kj}(x,\vec{E})$$

describes a model in which the kill intensity depends on allocation (α) and noise jamming $\vec{E}$, but deception due to E_{j*} is not considered.

5.5 DEFENSE SUPPRESSION

We now want to consider the possibility of using weapons to suppress or kill defense sites. We assume each penetrator of type j carries S_{jk} weapons to either kill or suppress defense units of type k. With an appropriate definition of penetrator subgroup of type j, it is assumed that each (j,k) combination may be categorized as "suppression" or "destruction." Thus, a given penetrator type j will not both suppress and destroy a defense of given type.

We assume that these weapons are expended at a rate proportional to $f_k(x)$, the density of type k defenses at penetration depth x. Moreover, it is assumed that these weapons are expended over $(0,Z_k)$. Should one want to use these weapons over (Y_k,Z_k) where $0 < Y_k < Z_k$ then we replace k type defenses by two identical defenses k_1 and k_2 where

$$f_{k_1}(x) = f_k(x) \qquad\qquad 0 \leq x \leq Y_k$$

$$= 0 \qquad\qquad Y_k < x$$

$$f_{k_2}(x) = 0 \qquad\qquad 0 \leq x \leq Y_k$$

$$= f_k(x) \qquad\qquad Y_k < x$$

Now $S_{jk_1} = 0$ and $S_{jk_2} = S_{jk}$ with the weapons allocated over $(0, Z_k)$. Since $f_{k_2}(x) = 0$ for $0 \le x \le Y_k$, the S_{jk} weapons per aircraft are allocated to k type defenses over (Y_k, Z_k).

Clearly a suppression weapon must be used at some point before x to be effective at x. We shall make the simplifying assumptions that the number of penetrators alive at x approximates those alive at a position to launch suppression weapons at defenses at x. Based on the expected number of surviving defense sites, $f_k(x, S)$, the density of sites after suppression, satisfies

$$2 \int_0^x R_k(h(y)) f_k(y, S) \, dy = 2 \int_0^x R_k(h(y)) f_k(y) \, dy$$

$$- 2 \int_0^x \sum_j \frac{S_{jk}\left(1 - (1 - P_{Sjk})^{r(y)}\right)}{r(y) N_k(Z_k)} E_j(y) R_k(h(y)) f_k(y) \, dy$$

where $r(y)$ is the smallest integer such that the expression on the right is non-negative, $R_k(h)$ is a nominal radius of action for the attackers against defense type k with the constraint

$$R_k(h(y)) \ge \max_j R_{kj}(h(y))$$

So that the attackers attempt to suppress all sites that can have an effect on the penetration

$$N_k(Z_k) = 2 \int_0^{Z_k} R_k(h(y)) f_k(y) \, dy$$

This result follows from the assumption that the rate of weapon expenditure is proportional to $f_k(x)$ and the requirement that all weapons be expended by the time Z_k is reached. The smaller $R_k(h(y))$, the more effective the penetration. Thus, if ECM is utilized

$$R_k(h(x)) \ge \max_j R_{kj}(x, \vec{E})$$

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and defense suppression together with ECM can have a beneficial joint effect.

Taking the derivative with respect to x yields

$$f_k(x,S) = f_k(x)\left[1 - \sum_j \frac{S_{jk}\left(1-(1-P_{Sjk})^{r(x)}\right)E_j(x)}{r(x)N_k(Z_k)}\right]$$
$$0 \leq x \leq Z_k$$

$$= f_k(x)' \qquad\qquad Z_k \leq x$$

where $r(x)$ is the smallest integer such that the expression in brackets is positive. In most instances we can set $r = 1$ and then

$$f_k(x,S) = \max\left[0, f_k(x)\left[1 - \sum_j \frac{S_{jk}P_{Sjk}E_j(x)}{N_k(Z_k)}\right]\right]$$
$$0 \leq x \leq Z_k$$

$$= f_k(x) \qquad\qquad Z_k \leq x$$

Now

$$\lambda_{kj}(x,\vec{E},S) = 2R_{kj}(h(x),\vec{E})M_{kj}(h(x),v(x),\vec{E})$$

$$\times \bar{P}_{Kkj}(h(x),v(x),\vec{E})f_k(x,S)$$

and

$$\lambda_{kj\alpha}(x,\vec{E},S) = \bar{\alpha}_{kj}(x,E_{j*})\lambda_{kj}(x,\vec{E},S)$$

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5.6 SELF-DEFENSE

The situation in self-defense is much the same as in defense suppression. In this case the weapons that j type penetrators allocate for defense against the weapons that k type defenses utilize must be used against these types of weapons.

If $W_{kj}(h,v)$ is the expected number of weapons a type k defense can get off at a type j penetrator, then the total number of weapons expended by k type defenses against j type penetrators is

$$W_{kj\alpha}(x,\vec{E},E_{j*},S) = 2 \int_0^x M_{kj}(h(y),v(y),\vec{E}) R_{kj}(h(y),\vec{E}) \bar{\alpha}_{kj}(y,\vec{E}_{j*}) f_k(y,S) dy$$

if jamming, deception, suppression and defense allocation are all considered.

The penetrators of type j attempt to counter these weapons with d_{jk} defense weapons per j type penetrator with kill probability P_{djk} per defense weapon. We assume that the j type penetrators expend these weapons at a rate proportional to the density of weapons defense type k launches at penetrator type j. Thus, these weapons are expended at a rate proportional to $dW_{kj\alpha}(x,\vec{E},E_{j*},S)/dx$ over distance $(0,Z_{jk})$. The cumulative number of surviving weapons due to self-defense is

$$W_{kj\alpha}(x,\vec{E},E_{j*},S,D) \equiv 2 \int_0^x M_{kj}(h(y),v(y),\vec{E},D) R_{kj}(h(y),\vec{E}) \bar{\alpha}_{kj}(y,\vec{E}_{j*}) f_k(y,S) dy$$

$$= \int_0^x \max\left[0, \frac{dW_{kj\alpha}(y,\vec{E},E_{j*},S)}{dy}\left(1 - \frac{d_{jk}P_{djk}E_j(y)}{N_{jk}(Z_{jk})}\right)\right] dy$$

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It follows that

$$\frac{dW_{kj\alpha}(x,\vec{E},E_{j*},S,D)}{dx} = \max\left[0, \frac{dW_{kj\alpha}(x,\vec{E},E_{j*},S)}{dx}\left(1 - \frac{d_{jk}P_{djk}E_j(x)}{N_{jk}(Z_{jk})}\right)\right]$$

where

$$N_{jk}(Z_{jk}) = \int_0^{Z_{jk}} \frac{dW_{kj\alpha}(x,\vec{E},E_{j*},S)}{dx}\,dx$$

$$= W_{kj\alpha}(Z_{jk},\vec{E},E_{j*},S)$$

It follows that

$$M_{kj}(h(x),v(x),\vec{E},D) = \max\left[0,M_{kj}(h(x),v(x),\vec{E})\left(1 - \frac{d_{jk}P_{djk}E_j(x)}{N_{jk}(Z_{jk})}\right)\right]$$

$$0 \leq x \leq Z_{jk}$$

$$= M_{kj}(h(x),v(x),\vec{E}) \qquad\qquad Z_{jk} < x$$

Now

$$\lambda_{kj}(x,\vec{E},S,D) = 2R_{kj}(h(x),\vec{E})M_{kj}(h(x),v(x),\vec{E},D)$$

$$\times \bar{P}_{Kkj}(h(x),v(x),\vec{E})f_k(x,S)$$

and

$$\lambda_{kj\alpha}(x,\vec{E},S,D) = \lambda_{kj}(x,\vec{E},S,D)\bar{\alpha}_{kj}(x,\vec{E},E_{j*})$$

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5.7 STANDOFF WEAPONS

We assume that one particular subgroup, j_0, say, intends to launch standoff weapons at D_0 for impact at $r_0 \geq D_0$. If each element of subgroup j_0 carries w_{j_0} standoff weapons, then the probability that $w_{j_0}i_{j_0}$ standoff weapons are launched at D_0 is $Q_{i_{j_0}}(D_0)$. The probability distribution of the number of standoff weapons surviving to r_0 is

$$Q_i(r_0) = \sum_{i_{j_0} = \left[i/w_{j_0}\right]^*}^{N_{j_0}} Q_{i_{j_0}}(D_0) Q_{i|w_{j_0}i_{j_0}}(D_0, r_0)$$

where $\left[i/w_{j_0}\right]^*$ is the smallest integer $\geq i/w_{j_0}$. Providing the model of 3.1 may be used,

$$Q_{i|w_{j_0}i_{j_0}}(D_0, r_0) = \frac{[H(D_0, r_0)]^{w_{j_0}i_{j_0}-i}}{(w_{j_0}i_{j_0} - i)!} e^{-H(D_0, r_0)}$$

$$Q_{0|w_{j_0}i_{j_0}}(D_0, r_0) = 1 - \sum_{i=1}^{w_{j_0}i_{j_0}} Q_{i|w_{j_0}i_{j_0}}(D_0, r_0)$$

$$H(D_0, r_0) = \int_{D_0}^{r_0} \lambda_m(y)\, dy$$

where $\lambda_m(y)$ is the kill intensity due to all defenses against standoff missiles at position y. It is understood that $\lambda_m(y)$ is obtained using concepts developed in Section 4.0.

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Appendix A

A NUMERICAL COMPARISON OF POISSON AND BINOMIAL DISTRIBUTIONS

In this appendix we compare the truncated Poisson with the binomial distributions associated with SLS and MSE firing doctrines. For a fixed H, there are a number of combinations of M and P_k that may be used. The binomial results depend on both M and P_k and it is important to see how the truncated Poisson results compare with more acceptable binomial results.

Tables A-1 through A-7 allow one to make this comparison for a range of N, M, P_k and H. Generally speaking the truncated Poisson results lie between those for SLS and those for MSE. Since most penetrations will involve a mixture of MSE and SLS firing doctrines, this is an encouraging result. Stated differently, firing doctrine can produce a larger difference in results for binomial models than the difference between a truncated Poisson and either binomial model.

TABLE A-1. Comparison of Binomials and Truncated Poisson for H = 0.1

$P_K(M)$ ╲ N	1	2	5	10	20
Expected Kills for the Binomials SLS/MSE					
.005 (20)	.095/(.095)	.100/(.098)	.100/(.099)	.100/(.100)	.100/(.100)
.01 (10)	.096/(.096)	.100/(.098)	.100/(.100)	.100/(.100)	.100/(.100)
.025 (4)	.096/(.096)	.100/(.099)	.100/(.100)	.100/(.100)	.100/(.100)
.05 (2)	.098/(.098)	.100/(.100)	.100/(.100)	.100/(.100)	.100/(.100)
.10 (1)	.100/(.100)	.100/(.100)	.100/(.100)	.100/(.100)	.100/(.100)
Properties of Truncated Poisson					
Expected number killed	.095	.100	.100	.100	.100
For SLS: Max % error	5.0	0	0	0	0
Avg % error	2.0	0	0	0	0
For MSE: Max % error	5.0	-2.0	-1.0	0	0
Avg % error	2.0	-1.0	-0.2	0	0
Avg % error for both SLS and MSE	2.0	-0.5	-0.1	0	0

TABLE A-2. Comparison of Binomials and Truncated Poisson for H = 0.25.

$P_K(M)$ N	1	2	5	10	20
Expected Kills for the Binomials SLS/MSE					
.005 (50)	.222/(.222)	.248/(.236)	.25/(.244)	.25/(.248)	.25/(.249)
.01 (25)	.222/(.222)	.248/(.236)	.25/(.245)	.25/(.248)	.25/(.25)
.025 (10)	.224/(.224)	.248/(.238)	.25/(.247)	.25/(.25)	.25/(.25)
.05 (5)	.226/(.226)	.249/(.241)	.25/(.25)	.25/(.25)	.25/(.25)
.10 (2.5)	.232/(.232)	.250/(.247)	.25/(.25)	.25/(.25)	.25/(.25)
.25 (1)	.250/(.250)	.250/(.250)	.25/(.25)	.25/(.25)	.25/(.25)
Properties of Truncated Poisson					
Expected number killed	.221	.248	.25	.25	.25
For SLS: Max % error	11.6	0.8	0	0	0
Avg % error	3.5	0.4	0	0	0
For MSE: Max % error	11.6	-5.1	-2.5	-0.8	-0.4
Avg % error	3.5	-2.8	-1.0	-0.2	0
Avg % error for both SLS and MSE	3.5	-1.2	-0.5	-0.1	0

TABLE A-3. Comparison of Binomials and Truncated Poisson for H = 0.50.

$P_K(M)$ N	1	2	5	10	20
Expected Kills for the Binomials SLS/MSE					
.005 (100)	.394/(.394)	.484/(.443)	.5/(.477)	.5/(.489)	.5/(.495)
.01 (50)	.395/(.395)	.484/(.444)	.5/(.478)	.5/(.490)	.5/(.496)
.025 (20)	.397/(.397)	.486/(.447)	.5/(.482)	.5/(.494)	.5/(.5)
.05 (10)	.401/(.401)	.487/(.450)	.5/(.49)	.5/(.5)	.5/(.5)
.10 (5)	.41/(.41)	.49/(.46)	.5/(.5)	.5/(.5)	.5/(.5)
.25 (2)	.437/(.437)	.50/(.50)	.5/(.5)	.5/(.5)	.5/(.5)
.50 (1)	.5/(.5)	.5/(.5)	.5/(.5)	.5/(.5)	.5/(.5)
Properties of Truncated Poisson					
Expected number killed	.393	.484	.5	.5	.5
For SLS: Max % error	21.4	3.2	0	0	0
Avg % error	5.6	1.2	0	0	0
For MSE: Max % error	21.4	-9.2	-4.8	-2.2	-1.0
Avg % error	5.6	-4.6	-2.2	-0.8	-0.2
Avg % error for both SLS and MSE	5.6	-1.7	-1.1	-0.4	-0.1

TABLE A-4. Comparison of Binomials and Truncated Poisson for H = 1.

$P_K(M)$ ＼ N	1	2	5	10	20
Expected Kills for the Binomials SLS/MSE					
.005 (200)	.633/(.633)	.897/(.788)	.999/(.908)	1.00/(.954)	1.00/(.978)
.01 (100)	.634/(.634)	.898/(.790)	.999/(.910)	1.00/(.956)	1.00/(.98)
.025 (40)	.637/(.637)	.901/(.795)	1.0/(.917)	1.0/(.963)	1.0/(.987)
.05 (20)	.642/(.642)	.906/(.802)	1.0/(.927)	1.0/(.975)	1.0/(1.0)
.10 (10)	.651/(.651)	.915/(.819)	1.0/(.95)	1.0/(1.0)	1.0/(1.0)
.25 (4)	.684/(.684)	.945/(.875)	1.0/(1.0)	1.0/(1.0)	1.0/(1.0)
.50 (2)	.75/(.75)	1.0/(1.0)	1.0/(1.0)	1.0/(1.0)	1.0/(1.0)
Properties of Truncated Poisson					
Expected number killed	.632	.896	.999	1.0	1.0
For SLS: Max % error	36.8	10.4	0.1	0	0
Avg % error	8.2	3.8	0.1	0	0
For MSE: Max % error	36.8	-13.7	-10.0	-4.8	-2.2
Avg % error	8.2	-5.4	-5.1	-2.0	-0.2
Avg % error for both SLS and MSE	8.2	-0.8	-2.5	-1.0	-0.1

TABLE A-5. Comparison of Binomials and Truncated Poisson for H = 2.

$P_K(M)$ ＼ N	1	2	5	10	20
Expected Kills for the Binomials SLS/MSE					
.05 (40)	.87/(.87)	1.47/(1.28)	1.98/(1.68)	2.00/(1.86)	2.00/(1.95)
.10 (20)	.88/(.88)	1.49/(1.30)	1.99/(1.72)	2.00/(1.90)	2.00/(2.00)
.25 (8)	.90/(.90)	1.53/(1.37)	2.00/(1.84)	2.00/(2.00)	2.00/(2.00)
.50 (4)	.94/(.94)	1.62/(1.50)	2.00/(2.00)	2.00/(2.00)	2.00/(2.00)
.67 (3)	.96/(.96)	1.70/(1.62)	2.0/(2.0)	2.00/(2.00)	2.00/(2.00)
1.0 (2)	1.00/(1.00)	2.00/(2.00)	2.0/(2.0)	2.00/(2.00)	2.00/(2.00)
Properties of Truncated Poisson					
Expected number killed	.87	1.46	1.98	2.00	2.00
For SLS: Max % error	13.0	27.0	1.0	0	0
Avg % error	5.7	9.7	0.8	0	0
For MSE: Max % error	13.0	27.0	-17.8	-7.5	-2.6
Avg % error	5.7	1.1	-6.3	-2.1	-0.4
Avg % error for both SLS and MSE	5.7	5.4	-2.7	-1.1	-0.2

TABLE A-6. Comparison of Binomials and Truncated Poisson for H = 5.

$P_K(M)$ ⟍ N	1	2	5	10	20
Expected Kills for the Binomials SLS/MSE					
.05 (100)	.99/(.99)	1.96/(1.85)	4.14/(3.21)	4.98/(4.01)	5/(4.52)
.10 (50)	1.00/(1.00)	1.97/(1.86)	4.17/(3.25)	4.99/(4.10)	5/(4.63)
.25 (20)	1.00/(1.00)	1.98/(1.89)	4.24/(3.42)	4.99/(4.38)	5/(5.0)
.50 (10)	1.00/(1.00)	1.99/(1.94)	4.38/(3.75)	5.00/(5.00)	5/(5.0)
.67 (7.5)	1.00/(1.00)	2.00/(1.97)	4.50/(4.04)	5.00/(5.00)	5/(5.0)
.83 (6)	1.00/(1.00)	2.00/(1.99)	4.67/(4.42)	5.00/(5.00)	5/(5.0)
1.00 (5)	1.00/(1.00)	2.00/(2.00)	5.00/(5.00)	5.00/(5.00)	5/(5.0)
Properties of Truncated Poisson					
Expected number killed	.99	1.95	4.12	4.98	5
For SLS: Max % error	1.0	2.5	17.6	0.4	0
Avg % error	1.0	1.8	6.8	0.3	0
For MSE: Max % error	1.0	-5.4	-28.3	-24.2	-10.6
Avg % error	1.0	-1.2	-9.0	-8.2	-2.6
Avg % error for both SLS and MSE	1.0	0.3	-1.1	-4.0	-1.3

TABLE A-7. Comparison of Binomials and Truncated Poisson for H = 10.

$P_K(M)$ ⟍ N	1	2	5	10	20
Expected Kills for the Binomials SLS/MSE					
.05 (200)	1.00/(1.00)	2/(1.99)	4.96/(4.36)	8.78/(6.42)	10/(8.02)
.10 (100)	1.00/(1.00)	2/(1.99)	4.97/(4.39)	8.81/(6.51)	10/(8.19)
.25 (40)	1.00/(1.00)	2/(1.99)	4.98/(4.50)	8.92/(6.84)	10/(8.75)
.50 (20)	1.00/(1.00)	2/(1.99)	4.99/(4.69)	9.12/(7.50)	10/(10)
.67 (15)	1.00/(1.00)	2/(2.00)	5.00/(4.82)	9.28/(8.08)	10/(10)
.83 (12)	1.00/(1.00)	2/(2.00)	5.00/(4.93)	9.89/(8.84)	10/(10)
1.00 (10)	1.00/(1.00)	2/(2.00)	5.00/(5.00)	10.0/(10.0)	10/(10)
Properties of Truncated Poisson					
Expected number killed	1	2	4.96	8.75	10
For SLS: Max % error	0	0	0.8	12.5	0
Avg % error	0	0	0.5	5.2	0
For MSE: Max % error	0	-0.5	-13.8	-36.3	-24.7
Avg % error	0	-0.3	-6.5	-15.7	-8.7
Avg % error for both SLS and MSE	0	-0.1	-3.0	-5.3	-4.4

Tables A-8 through A-14 were constructed by considering that the results of any penetration are indeed a mixture of MSE and SLS results. Generally speaking, when M/N is large one would expect SLS to predominate. We consider a weighting of the form

$$\phi(\alpha) = e^{-\alpha M/N} E(MSE) + \left(1 - e^{-\alpha M/N}\right) E(SLS)$$

where α is chosen so that $e^{-\alpha M/N} = 1/2$ when M/N is as close to 2 as is possible. Roughly, we assume MSE is as likely as SLS when $M/N \cong 2$. Other weightings could be chosen, but the point is to compare the results of the truncated Poisson with some reasonable combination of SLS and MSE results.

The truncated Poisson gives results fully comparable with classical binominal results when the uncertainty of firing doctrine is taken into account. The approximation is worse when

$$1/2 \ N \leq H \leq 2N$$

TABLE A-8. Comparison of Binomials and Truncated Poisson for H = 0.1.

$P_K(M)$ \ N	1	2	5	10	20
Expected Kills for the Weighted Binomials SLS/MSE					
.005 (20)	.095	.100	.100	.100	.100
.01 (10)	.096	.100	.100	.100	.100
.025 (4)	.096	.100	.100	.100	.100
.05 (2)	.098	.100	.100	.100	.100
.10 (1)	.100	.100	.100	.100	.100
Properties of Truncated Poisson					
Expected number killed	.095	.100	.100	.100	.100
Max % error	5.0 at 0.1	0	0	0	0
Max for $P_K \leq 0.5$	5.0 at 0.1	0	0	0	0
Min % error	0 at 0.005	0	0	0	0
Min for $P_K \leq 0.5$	0 at 0.005	0	0	0	0
Avg % error $P_K \leq 0.5$	2.0	0	0	0	0

TABLE A-9. Comparison of Binomials and Truncated Poisson for H = 0.25.

$P_K(M)$ ╲ N	1	2	5	10	20
Expected Kills for the Weighted Binomials SLS/MSE					
.005 (50)	.222	.248	.250	.250	.25
.01 (25)	.222	.248	.249	.249	.25
.025 (10)	.224	.246	.248	.25	.25
.05 (5)	.226	.245	.25	.25	.25
.10 (2.5)	.232	.248	.25	.25	.25
.25 (1)	.250	.250	.25	.25	.25
Properties of Truncated Poisson					
Expected number killed	.221	.248	.25	.25	.25
Max % error Max for $P_K \leq 0.5$	11.6 at 0.25 11.6 at 0.25	0.8 at 0.25 0.8 at 0.25	0 0	0 0	0 0
Min % error Min for $P_K \leq 0.5$	0.4 at 0.005 0.4 at 0.005	-1.2 at 0.05 -1.2 at 0.05	-0.8 at 0.025 -0.8 at 0.025	-0.4 at 0.01 -0.4 at 0.01	0 0
Avg % error $P_K \leq 0.5$	3.5	-0.2	-0.2	-0.1	0

TABLE A-10. Comparison of Binomials and Truncated Poisson for H = 0.5.

$P_K(M)$ ╲ N	1	2	5	10	20
Expected Kills for the Weighted Binomials SLS/MSE					
.005 (100)	.394	.484	.5	.5	.499
.01 (50)	.395	.484	.499	.498	.498
.025 (20)	.397	.484	.496	.497	.5
.05 (10)	.401	.478	.495	.5	.5
.10 (5)	.410	.475	.5	.5	.5
.25 (2)	.437	.5	.5	.5	.5
.50 (1)	.5	.5	.5	.5	.5
Properties of Truncated Poisson					
Expected number killed	.393	.484	.5	.5	.5
Max % error Max for $P_K \leq 0.5$	21.4 at 0.5 21.4 at 0.5	3.2 at 0.5 3.2 at 0.5	0 0	0 0	0 0
Min % error Min for $P_K \leq 0.5$	0.2 at 0.005 0.2 at 0.005	-1.9 at 0.10 -1.9 at 0.10	-1.0 at 0.05 -1.0 at 0.05	-0.6 at 0.025 -0.6 at 0.025	-0.4 at 0.01 -0.4 at 0.01
Avg % error $P_K \leq 5$	5.6	0.5	-0.3	-0.1	-0.1

TABLE A-11. Comparison of Binomials and Truncated Poisson for $H = 1$.

$P_K(M)$ \ N	1	2	5	10	20
Expected Kills for the Weighted Binomials SLS/MSE					
.005 (200)	.633	.897	.999	1.000	.999
.01 (100)	.634	.898	.999	.999	.996
.025 (40)	.637	.901	.995	.991	.994
.05 (20)	.642	.903	.982	.988	1.000
.10 (10)	.651	.898	.975	1.000	1.000
.25 (4)	.684	.910	1.000	1.000	1.000
.50 (2)	.750	1.000	1.000	1.000	1.000
Properties of Truncated Poisson					
Expected number killed	.632	.896	.999	1.000	1.000
Max % error Max for $P_K \le 0.5$	36.8 at 0.5 36.8 at 0.5	10.4 at 0.5 10.4 at 0.5	0.1 at 0.5 0.1 at 0.5	0 0	0 0
Min % error Min for $P_K \le 0.5$	0.2 at 0.005 0.2 at 0.005	0.1 at 0.005 0.1 at 0.005	-2.5 at 0.10 -2.5 at 0.10	-1.2 at 0.05 -1.2 at 0.05	-0.6 at 0.025 -0.6 at 0.025
Avg % error $P_K \le 0.5$	8.2	2.0	-0.6	-0.3	-0.2

TABLE A-12. Comparison of Binomials and Truncated Poisson for $H = 2$.

$P_K(M)$ \ N	1	2	5	10	20
Expected Kills for the Weighted Binomials SLS/MSE					
.05 (40)	.87	1.47	1.97	1.96	1.98
.10 (20)	.88	1.48	1.94	1.95	2.00
.25 (8)	.90	1.49	1.92	2.00	2.00
.50 (4)	.94	1.56	2.00	2.00	2.00
.67 (3)	.96	1.65	2.00	2.00	2.00
1.0 (2)	1.00	2.00	2.00	2.00	2.00
Properties of Truncated Poisson					
Expected number killed	.87	1.46	1.58	2.00	2.00
Max % error Max for $P_K \le 0.5$	13.0 at 1.00 7.4 at 0.5	27.0 at 1.00 6.4 at 0.5	1.0 at 1.00 0 at 0.5	0 0	0 0
Min % error Min for $P_K \le 0.5$	0 at 0.05 0 at 0.05	0.7 at 0.05 0.7 at 0.05	-3.1 at 0.25 -3.1 at 0.25	-2.5 at 0.10 -2.5 at 0.10	-1.0 at 0.05 -1.0 at 0.05
Avg % error $P_K \le 0.5$	3.2	2.6	-1.2	-1.2	-0.2

TABLE A-13. Comparison of Binomials and Truncated Poisson for H = 5.

$P_K(M)$ ＼ N	1	2	5	10	20
Expected Kills for the Weighted Binomials SLS/MSE					
.05 (100)	.99	1.96	4.14	4.95	4.88
.10 (50)	1.00	1.97	4.14	4.83	4.82
.25 (20)	1.00	1.97	4.04	4.68	5.00
.50 (10)	1.00	1.98	4.06	5.00	5.00
.67 (7.5)	1.00	1.99	4.23	5.00	5.00
.83 (6)	1.00	2.00	4.50	5.00	5.00
1.00 (5)	1.00	2.00	5.00	5.00	5.00
Properties of Truncated Poisson					
Expected number killed	.99	1.95	4.12	4.98	5.00
Max % error Max for $P_K \leq 0.5$	1.0 at 1.00 1.0 at 0.50	2.5 at 1.00 1.5 at 0.5	17.6 at 1.00 0.5 at 0.10	0.4 at 1.00 0.4 at 0.5	0 0
Min % error Min for $P_K \leq 0.5$	0 at 0.05 0 at 0.05	0.5 at 0.05 0.5 at 0.05	-2.0 at 0.25 -2.0 at 0.25	-6.4 at 0.25 -6.4 at 0.25	-3.7 at 0.10 -3.7 at 0.10
Avg % error $P_K \leq 0.5$	0.8	1.0	-0.6	-2.4	-1.5

TABLE A-14. Comparison of Binomials and Truncated Poisson for H = 10.

$P_K(M)$ ＼ N	1	2	5	10	20
Expected Kills for the Weighted Binomials SLS/MSE					
.05 (200)	1.00	2.00	4.96	8.78	9.94
.10 (100)	1.00	2.00	4.97	8.74	9.68
.25 (40)	1.00	2.00	4.95	8.40	9.38
.50 (20)	1.00	2.00	4.92	8.31	10.00
.67 (15)	1.00	2.00	4.94	8.57	10.00
.83 (12)	1.00	2.00	4.97	9.20	10.00
1.00 (10)	1.00	2.00	5.00	10.00	10.00
Properties of Truncated Poisson					
Expected number killed	1.00	2.00	4.96	8.75	10.00
Max % error Max for $P_K \leq 0.5$	0 0	0 0	0.8 at 1.0 0.2 at 0.10	12.5 at 1.00 0.3 at 0.05	0 0
Min % error Min for $P_K \leq 0.5$	0 0	0 0	-0.8 at 0.50 -0.8 at 0.50	-5.3 at 0.5 -5.3 at 0.5	-6.6 at 0.25 -6.6 at 0.25
Avg % error $P_K \leq 0.5$	0	0	-0.2	-2.3	-2.6

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